Solutions for Final Exam

Instructions: These are solutions: you don't need instructions

Solution to Problem 1

1. We recall the Frenet-Serret equations

$$\begin{aligned} \mathbf{t}' &= \quad k\mathbf{n} \\ \mathbf{n}' &= -k\mathbf{t} \quad - \tau\mathbf{b} \\ \mathbf{t}' &= \quad \tau\mathbf{n} \end{aligned}$$

and we observe by direct calculation that the right-hand side of the Frenet-Serret equations are equal to the right-hand side of the Darboux equations, thus demonstrating the equivalence between these two sets of equations.

$$\begin{aligned} \mathbf{t} \wedge \omega &= -k\mathbf{t} \wedge \mathbf{b} = k\mathbf{n} \\ \mathbf{n} \wedge \omega &= \tau\mathbf{n} \wedge \mathbf{t} - k\mathbf{n} \wedge \mathbf{b} = -\tau\mathbf{b} - k\mathbf{t} \\ \mathbf{b} \wedge \omega &= \tau\mathbf{b} \wedge \mathbf{t} = \tau\mathbf{n}. \end{aligned}$$

2. We compute the derivative

$$\begin{aligned} \omega' &= \tau'\mathbf{t} + \tau\mathbf{t}' - k'\mathbf{b} - k\mathbf{b}' \\ &= \tau'\mathbf{t} - k'\mathbf{b} + \tau\mathbf{t} \wedge \omega - k\mathbf{b} \wedge \omega \\ &= \tau'\mathbf{t} - k'\mathbf{b} + (\tau\mathbf{t} - k\mathbf{b}) \wedge \omega \\ &= \tau'\mathbf{t} - k'\mathbf{b} + \omega \wedge \omega \\ &= \tau'\mathbf{t} - k'\mathbf{b}. \end{aligned}$$

Since $\mathbf{t}$ and $\mathbf{b}$ are independent, ω' vanishes if and only if both k' and τ' vanish. It follows that ω is constant if and only if both k and τ are constant.

For the rest of the exercise we have that k and τ are constant.

3. We have $\boldsymbol{v} = \beta^{-1}\omega$, where β is constant. Therefore the previous question implies $\boldsymbol{v}' = 0$. Also, we have $\boldsymbol{u} = \boldsymbol{v} \wedge \mathbf{n} = \beta^{-1}\omega \wedge \mathbf{n} = -\beta^{-1}\mathbf{n}'$, by the first question. It follows that $\mathbf{n}' = -\beta\boldsymbol{u}$. Finally, $\boldsymbol{u}' = \boldsymbol{v}' \wedge \mathbf{n} + \boldsymbol{v} \wedge \mathbf{n}' = \boldsymbol{v} \wedge \mathbf{n}' = -\beta\boldsymbol{v} \wedge \boldsymbol{u} = \beta\mathbf{n}$.

4. Let $U(s) = \cos(\beta s)\boldsymbol{u}_0 + \sin(\beta s)\mathbf{n}_0$. Then $U(0) = \boldsymbol{u}_0$ and $U'(0) = \beta\mathbf{n}_0 = \boldsymbol{u}'(0)$. Since both U and $\boldsymbol{u}$ are solutions of the second order linear differential equation $u'' + \beta u = 0$, with the same initial conditions, we see that $u(s) = U(s)$ for $s \in \mathbb{R}$.

Alternate solution: We have that $\boldsymbol{v}(s) = \boldsymbol{v}_0$, a constant. This means that $\mathbf{n}$ and $\boldsymbol{u}$ always lie in the plane spanned by $\mathbf{n}_0$ and $\boldsymbol{u}_0$. Paying attention to our orientation we write

$$\mathbf{n}(s) = \cos(\theta(s))\mathbf{n}_0 + \sin(\theta(s))\boldsymbol{u}_0$$

and

$$\boldsymbol{u}(s) = -\sin(\theta(s))\mathbf{n}_0 + \cos(\theta(s))\boldsymbol{u}_0.$$

The fact that such a function $\theta(s)$ exists follows from the fact that $\mathbf{n}(s)$ is a unit-vector field along α. This is demonstrated by Lemma 1 on p. 250.

We find

$$\boldsymbol{u}' = -\theta'(\cos\theta\mathbf{n}_0 + \sin\theta\boldsymbol{u}_0) = \beta\mathbf{n}.$$

This implies that $\theta(s) = -\beta s + C$, for some constant $C \in \mathbb{R}$. The initial condition $\boldsymbol{u}(0) = \boldsymbol{u}_0$ implies $C = 0$. Therefore we have

$$\begin{aligned}\boldsymbol{u}(s) &= -\sin(-\beta s)\mathbf{n}_0 + \cos(-\beta s)\boldsymbol{u}_0 \\ &= \sin(\beta s)\mathbf{n}_0 + \cos(\beta s)\boldsymbol{u}_0,\end{aligned}$$

as desired, and also

$$\mathbf{n}(s) = \cos(\beta s)\mathbf{n}_0 - \sin(\beta s)\boldsymbol{u}_0.$$

5. Since $\mathbf{t}$ is orthogonal to $\mathbf{n}$, we may write $\mathbf{t} = a\boldsymbol{u} + b\boldsymbol{v}$. Then we observe that $b = \langle \boldsymbol{v}, \mathbf{t} \rangle = \beta^{-1}\langle \omega, \mathbf{t} \rangle = \beta^{-1}\tau$, which is a constant. Since $\mathbf{t}$ is a unit vector, we have $a^2 + \beta^{-2}\tau^2 = 1$, which implies that $a = \beta^{-1}k$. Therefore

$$\mathbf{t}(s) = \beta^{-1}(k\boldsymbol{u}(s) + \tau\boldsymbol{v}_0).$$

Integrating, we have

$$\begin{aligned}\alpha(s) &= \alpha(0) + \int_0^s \mathbf{t}(t)\,dt \\ &= \alpha(0) + \beta^{-1}k\int_0^s \cos(\beta t)\boldsymbol{u}_0 + \sin(\beta t)\mathbf{n}_0\,dt + \beta^{-1}\tau s\boldsymbol{v}_0 \\ &= \alpha(0) + \beta^{-2}k\Big(\sin(\beta s)\boldsymbol{u}_0 - \cos(\beta s)\mathbf{n}_0 + \mathbf{n}_0\Big) + \beta^{-1}\tau s\boldsymbol{v}_0 \\ &= (\alpha(0) + \beta^{-2}k\mathbf{n}_0) + \beta^{-2}k\Big(\sin(\beta s)\boldsymbol{u}_0 - \cos(\beta s)\mathbf{n}_0\Big) + \beta^{-1}\tau s\boldsymbol{v}_0\end{aligned}$$

Let $p = \alpha(0) + \beta^{-2}k\mathbf{n}_0$, and let

$$\boldsymbol{x}(\xi, \eta) = p + \xi\boldsymbol{v}_0 + r\big(\boldsymbol{u}_0\sin\eta - \mathbf{n}_0\cos\eta\big)$$

be a parameterisation of the cylinder described in the exercise. Then we observe that $\alpha(s) = \boldsymbol{x}(\beta^{-1}\tau s, \beta s)$ is a curve on the cylinder.

Solution to Problem 2

1. We compute

$$\begin{aligned} \boldsymbol{y}_u &= \boldsymbol{x}_u + \varepsilon N_u \\ \boldsymbol{y}_v &= \boldsymbol{x}_v + \varepsilon N_v, \end{aligned}$$

and we find

$$\boldsymbol{y}_u \wedge \boldsymbol{y}_v = \boldsymbol{x}_u \wedge \boldsymbol{x}_v + \varepsilon(\boldsymbol{x}_u \wedge N_v + N_u \wedge \boldsymbol{x}_v) + \varepsilon^2(N_u \wedge N_v).$$

Now, if $p = \boldsymbol{x}(q)$, then $N_u(q) = dN_p(\boldsymbol{x}_u)$, and $N_v(q) = dN_p(\boldsymbol{x}_v)$, so

$$N_u \wedge N_v = dN_p(\boldsymbol{x}_u) \wedge dN_p(\boldsymbol{x}_v) = \det(dN_p)\boldsymbol{x}_u \wedge \boldsymbol{x}_v = K(p)\boldsymbol{x}_u \wedge \boldsymbol{x}_v,$$

and we obtain the desired quadratic term.

For the term linear in ε, we express dN_p in matrix form with respect to the basis $\{\boldsymbol{x}_u, \boldsymbol{x}_v\}$, so

$$N_u = dN_p(\boldsymbol{x}_u) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = a_{11}\boldsymbol{x}_u + a_{21}\boldsymbol{x}_v,$$

and

$$N_v = dN_p(\boldsymbol{x}_v) = a_{12}\boldsymbol{x}_u + a_{22}\boldsymbol{x}_v.$$

Then

$$\begin{aligned} \boldsymbol{x}_u \wedge N_v + N_u \wedge \boldsymbol{x}_v &= (a_{22} + a_{11})\boldsymbol{x}_u \wedge \boldsymbol{x}_v \\ &= \operatorname{trace}(dN_p)\boldsymbol{x}_u \wedge \boldsymbol{x}_v \\ &= -2H\boldsymbol{x}_u \wedge \boldsymbol{x}_v, \end{aligned}$$

and we obtain the desired term linear in ε. Thus, putting it together, we have the desired expression:

$$\boldsymbol{y}_u \wedge \boldsymbol{y}_v = (1 - 2\varepsilon H + \varepsilon^2 K)\boldsymbol{x}_u \wedge \boldsymbol{x}_v.$$

2. Let $\tilde{N}$ be the normal to $\boldsymbol{y}$ oriented like $\boldsymbol{y}_u \wedge \boldsymbol{y}_v$. We observe that $\boldsymbol{y}_u \wedge \boldsymbol{y}_v = h(\boldsymbol{x}_u \wedge \boldsymbol{x}_v)$ with $h > 0$ implies that $\tilde{N} = N$. Then

$$K\boldsymbol{x}_u \wedge \boldsymbol{x}_v = N_u \wedge N_v = \tilde{N}_u \wedge \tilde{N}_v = \tilde{K}\boldsymbol{y}_u \wedge \boldsymbol{y}_v = \tilde{K}(1 - 2\varepsilon H + \varepsilon^2 K)\boldsymbol{x}_u \wedge \boldsymbol{x}_v,$$

and we obtain the desired expression for $\tilde{K}$.

Similarly, arguing as in the first part, we have

$$-2\tilde{H}\boldsymbol{y}_u \wedge \boldsymbol{y}_v = \boldsymbol{y}_u \wedge \tilde{N}_v + \tilde{N}_u \wedge \boldsymbol{y}_v,$$

which expands to

$$\begin{aligned} -2\tilde{H}(1 - 2\varepsilon H + \varepsilon^2 K)\boldsymbol{x}_u \wedge \boldsymbol{x}_v &= (\boldsymbol{x}_u + \varepsilon N_u) \wedge N_v + N_u \wedge (\boldsymbol{x}_v + \varepsilon N_v) \\ &= -2H\boldsymbol{x}_u \wedge \boldsymbol{x}_v + 2\varepsilon N_u \wedge N_v \\ &= -2(H - \varepsilon K)\boldsymbol{x}_u \wedge \boldsymbol{x}_v, \end{aligned}$$

and we obtain the desired equation for $\tilde{H}$.

Solution to Problem 3

1. The curve C is a meridian, and is therefore a geodesic of S (p. 255). An arc-length parameterisation is given by $\alpha(s) = (f(s), 0, g(s))$. Let w be the parallel vector field along α such that $w(s_0) = w_p$, and let φ be the angle between $\alpha'(s)$ and $w(s)$, so $\varphi(s_0) = \varphi_0$. Since α is a geodesic we have $\left[\frac{D\alpha'}{ds}\right] = 0$, and likewise, since w is parallel, $\left[\frac{Dw}{ds}\right] = 0$. Then it follows from Lemma 2 on p. 251 that $\frac{d\varphi}{ds} = 0$, and therefore $\varphi(s) = \varphi_0$ is a constant for all s.

2. Let $p = \boldsymbol{x}(u_0, v_0)$, and let $\beta \colon s \mapsto \boldsymbol{x}(u(s), v_0)$ be a unit-speed parameterisation of Γ. Since Γ is a circle of radius $r = f(v_0)$, it follows that $u(s) = s/r$.

Since $\boldsymbol{x}$ is an orthogonal parameterisation, we may employ Proposition 3 on p. 252, which implies that

$$\frac{d\varphi}{ds} = -\frac{1}{2\sqrt{EG}}\left\{G_u \frac{dv}{ds} - E_v \frac{du}{ds}\right\}. \tag{1}$$

In our parameterisation we have

$$\begin{aligned} \boldsymbol{x}_u(u, v) &= (-f(v)\sin u, f(v)\cos u, 0) \\ \boldsymbol{x}_v(u, v) &= (f'(v)\cos u, f'(v)\sin u, g'(v)), \end{aligned}$$

which implies that

$$2E = f^2 \qquad F = 0 \qquad G = (f')^2 + (g')^2 = 1.$$

In particular, $E_v = 2ff'$, and $G_u = 0$, and Equation (1) becomes

$$\frac{d\varphi}{ds} = \frac{1}{2f} 2ff'u' = f'(v_0)\frac{du}{ds} = \frac{f'(v_0)}{r} = \frac{\cos\vartheta_0}{r}.$$

Integrating, we find

$$\Delta\varphi = \int_0^{2\pi r} \frac{d\varphi}{ds}\, ds = 2\pi\cos\vartheta_0.$$

3. If Γ is a geodesic, then $\Delta\varphi = 0$. This is a direct consequence of Lemma 2 on p. 251, together with the definitions of parallel transport and of geodesics, and is not particular to our particular context. The converse assertion is not generally true, but it is in this case.

The previous question shows that $\Delta\varphi = 0$ if and only if $\cos\vartheta_0 = 0$. Since we have defined $\vartheta_0 \in (0, \pi)$, it follows that $\Delta\varphi = 0$ if and only if $\vartheta_0 = \frac{\pi}{2}$. This means that the tangent planes along Γ are parallel to the z-axis, and this is a necessary and sufficient condition for the parallel Γ to be a geodesic, as demonstrated on p. 256.